EHRENFEUCHT-FRAÎSSÉ GAMES

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SYNONYMS
Ehrenfeucht games, EF-games

DEFINITION
The Ehrenfeucht-Fraïssé game (EF-game, for short) is played by two players, usually called the spoiler and the duplicator (in the literature, the two players are sometimes also called Samson and Delilah or, simply, player I and player II). The board of the game consists of two structures \( A \) and \( B \) of the same vocabulary. The spoiler’s intention is to show a difference between the two structures, while the duplicator tries to make them look alike. The rules of the classical EF-game are as follows: The players play a certain number \( r \) of rounds. Each round \( i \) consists of two steps. First, the spoiler chooses either an element \( a_i \) in the universe of \( A \) or an element \( b_i \) in the universe of \( B \). Afterwards, the duplicator chooses an element in the other structure, i.e., she chooses an element \( b_i \) in the universe of \( B \) if the spoiler’s move was in \( A \), respectively, an element \( a_i \) in the universe of \( A \) if the spoiler’s move was in \( B \).

After \( r \) rounds, the game finishes with elements \( a_1, \ldots, a_r \) chosen in \( A \) and \( b_1, \ldots, b_r \) chosen in \( B \), and exactly one of the two players has won the game. Roughly speaking, the duplicator has won if and only if the structures \( A \) and \( B \), restricted to the elements chosen during the rounds of the game, are indistinguishable. To give a precise description of the winning condition let us assume, for simplicity, that the vocabulary of the structures \( A \) and \( B \) only contains relation symbols. Precisely, the duplicator has won the game if and only if the following two conditions are met: (1) for all \( i,j \in \{1, \ldots, r\} \), \( a_i = a_j \) iff \( b_i = b_j \), and (2) for each arity \( k \), each relation symbol \( R \) of arity \( k \) in the vocabulary, and all \( i_1, \ldots, i_k \in \{1, \ldots, r\} \), the tuple \( (a_{i_1}, \ldots, a_{i_k}) \) belongs to the interpretation of \( R \) in the structure \( A \) if and only if the tuple \( (b_{i_1}, \ldots, b_{i_k}) \) belongs to the interpretation of \( R \) in the structure \( B \). Since the game is finite, one of the two players must have a winning strategy, i.e., he or she can always win the game, no matter how the other player plays.

MAIN TEXT
EF-games are a tool for proving expressivity bounds for query languages. They were introduced by Ehrenfeucht [1] and Fraïssé [3]. The fundamental use of the game comes from the fact that it characterizes first-order logic as follows: The duplicator has a winning strategy in the \( r \)-round EF-game on two structures \( A \) and \( B \) of the same vocabulary if, and only if, \( A \) and \( B \) satisfy the same first-order sentences of quantifier rank at most \( r \) (recall that the quantifier rank of a first-order formula is the maximum nesting depth of quantifiers occurring in the formula). This is known as the Ehrenfeucht-Fraïssé Theorem, and it gives rise to the following methodology for proving inexpressibility results, i.e., for proving that certain Boolean queries cannot be expressed in first-order logic: To show that a Boolean query \( Q \) is not definable in first-order logic, it suffices to find, for each positive integer \( r \), two structures \( A_r \) and \( B_r \) such that (1) \( A_r \) satisfies query \( Q \), (2) \( B_r \) does not satisfy query \( Q \), and (3) the duplicator has a winning strategy in the \( r \)-round EF-game on \( A_r \) and \( B_r \).

Using this methodology, one can prove, for example, that none of the following queries is definable in first-order logic: “Does the given structure’s universe have even cardinality?”, “Is the given graph connected?”, “Is the given graph a tree?” (cf., e.g., the textbook [4]). In fact, the described methodology is the major tool available for proving inexpressibility results when restricting attention to finite structures. Applying it, however, requires finding a winning strategy for the duplicator in the EF-game, and this often is a non-trivial task that involves complicated combinatorial arguments. Fortunately,
techniques are known that simplify this task, among them a number of sufficient conditions (e.g., Hanf-locality and Gaifman-locality) that guarantee the existence of a winning strategy for the duplicator (see e.g. the survey [2] and the textbook [4]).

Variants of EF-games exist also for other logics than first-order logic, e.g., for finite variable logics and for monadic second-order logic (details can be found in the textbook [4]).

CROSS REFERENCE
Locality, Logical Structure, Expressiveness of Query Languages, First-Order Logic

REFERENCES